Lösningsskiss till tentamen i Matematisk analys, del 1
764G07, 2021-01-07.

1. Rita grafen till $f(x) = \dfrac{x-1}{(2+x)^2}$.

- $D_f = \{x :\ x \neq -2\}$

- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x-1}{(2+x)^2} = \lim\limits_{x \to -\infty} \dfrac{x\left(1-\frac{1}{x}\right)}{x^2\left(\frac{2}{x}+1\right)^2} = 0 \Rightarrow$

  $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x-1}{(2+x)^2} = \lim\limits_{x \to \infty} \dfrac{x\left(1-\frac{1}{x}\right)}{x^2\left(\frac{2}{x}+1\right)^2} = 0 \Rightarrow$ $y=0$ – vågrät asymptot

  $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{x-1}{(2+x)^2} = -\infty \Rightarrow$ $x=-2$ – lodrät asymptot.

  $\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \dfrac{x-1}{(2+x)^2} = -\infty \nearrow$

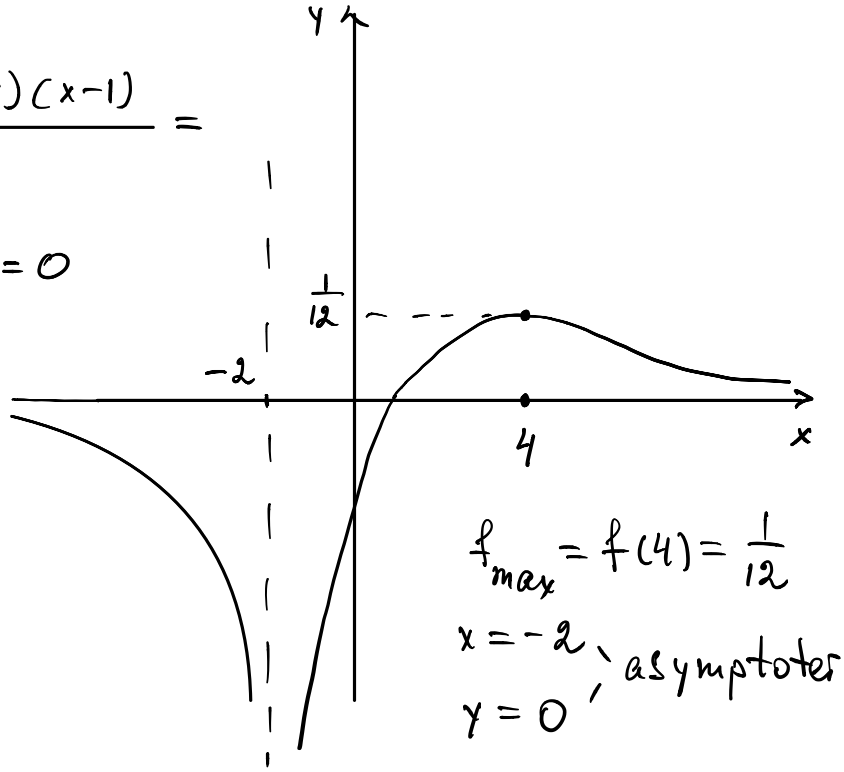
- $f'(x) = \dfrac{(2+x)^2 - 2(2+x)(x-1)}{(2+x)^4} =$

  $= \dfrac{2+x-2(x-1)}{(2+x)^3} = \dfrac{4-x}{(2+x)^3} = 0$

  $\Leftrightarrow \quad x = 4$

x		-2		4	
f'	$-$		$+$		$-$
f	$\searrow$		$\nearrow$	lok. max	$\searrow$

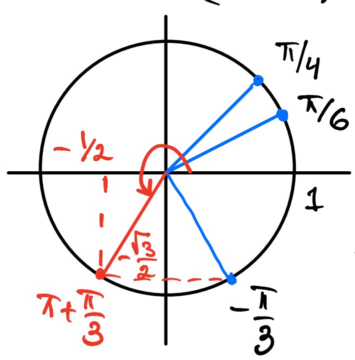
$f(4) = \dfrac{3}{36} = \dfrac{1}{12}$



$f_{max} = f(4) = \dfrac{1}{12}$

$x=-2$, $y=0$ asymptoter

2. $z = \dfrac{(1+i)^6}{(\sqrt{3}+i)^3(1-i\sqrt{3})}$



$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2e^{i\frac{\pi}{6}}$

$1-i\sqrt{3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2e^{-i\frac{\pi}{3}}$

$z = \dfrac{(\sqrt{2})^6 e^{i\frac{\pi}{4}\cdot 6}}{2^3 e^{i\frac{\pi}{6}\cdot 3}\cdot 2e^{-i\frac{\pi}{3}}} =$

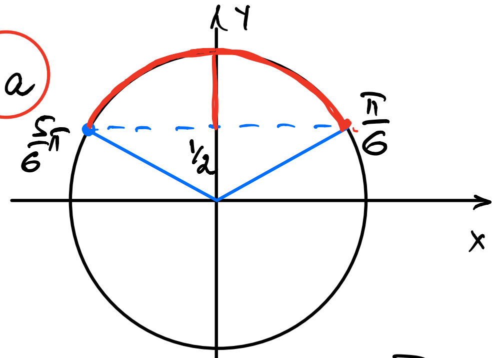
$= \frac{1}{2} e^{i\left(\frac{3}{2}\pi - \frac{\pi}{2} + \frac{\pi}{3}\right)} = \frac{1}{2} e^{i\left(\pi + \frac{\pi}{3}\right)} = \frac{1}{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -\frac{1}{4} - i\frac{\sqrt{3}}{4}$.

Svar: $\operatorname{Re} z = -\frac{1}{4}$, $\operatorname{Im} z = -\frac{\sqrt{3}}{4}$, $|z| = \frac{1}{2}$,

$\arg z = \frac{4}{3}\pi + 2\pi n$, $n \in \mathbb{Z}$.

3a



$$\sin(3x) > \frac{1}{2}, \quad 0 \le x \le \frac{\pi}{2}.$$

$$\frac{\pi}{6} + 2\pi n < 3x < \frac{5}{6}\pi + 2\pi n, \quad n \in \mathbb{Z}$$

$$\frac{\pi}{18} + \frac{2}{3}\pi n < x < \frac{5}{18}\pi + \frac{2}{3}\pi n$$

Notera att $\frac{\pi}{18} + \frac{2}{3}\pi = \frac{1+12}{18}\pi = \frac{13}{18}\pi > \frac{\pi}{2}$ samt

$$\frac{5}{18}\pi - \frac{2}{3}\pi = \frac{5-12}{18}\pi = -\frac{7}{18}\pi < 0.$$

Därför är lösningen $\frac{\pi}{18} < x < \frac{5}{18}\pi$.

3b $\quad 2\ln(4-x) = \ln(11-2x) \qquad (1)$

D_f: $\begin{cases} 4-x>0 \\ 11-2x>0 \end{cases} \Leftrightarrow \begin{cases} x<4 \\ x<11/2 \end{cases} \Leftrightarrow x<4$

(1) $\Leftrightarrow \begin{cases} \ln(4-x)^2 = \ln(11-2x) \\ x<4 \end{cases}$ / ln är injektiv / $\Leftrightarrow \begin{cases} (4-x)^2 = 11-2x \\ x<4 \end{cases}$

$\Rightarrow 16-8x+x^2 = 11-2x \Rightarrow x^2-6x+5=0 \Rightarrow (x-1)(x-5)=0$

$\Rightarrow x=1$, $x=5$ - falsk rot ty vi söker $x<4$.

Svar: a) $\frac{\pi}{18} < x < \frac{5}{18}\pi$, b) $x=1$

4a

$$\lim_{x\to 1} \frac{x^3-x^2-x+1}{x^3-3x+2} = \left[\frac{0}{0}\right] = \lim_{x\to 1} \frac{x^2(x-1)-(x-1)}{(x-1)(x^2+x-2)} =$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2-1)}{(x-1)(x^2+x-2)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x+2)} =$$

$$= \lim_{x\to 1} \frac{x+1}{x+2} = \frac{2}{3}$$

$$\begin{array}{r|l} x^2+x-2 & \\ \hline x^3-3x+2 & x-1 \\ -(x^3-x^2) & \\ \hline x^2-3x+2 & \\ -(x^2-x) & \\ \hline -2x+2 & \\ -(-2x+2) & \\ \hline 0 & \end{array}$$

4b

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x+2}-x\right) = [\infty-\infty] = \lim_{x\to\infty} \frac{(x^2+3x+2)-x^2}{\sqrt{x^2+3x+2}+x}$$

$$= \lim_{x\to\infty} \frac{3x+2}{\sqrt{x^2+3x+2}+x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{x(3+2/x)}{x\left(\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}+1\right)} = \frac{3}{2}$$

4c $\lim\limits_{x\to 2} \dfrac{1-\cos(2x-4)}{(x-2)\ln(3x-5)} \Bigg/ \begin{matrix} t = x-2 \\ t\to 0 \text{ då } x\to 2 \\ 3x-5 = 3(x-2)+1 \end{matrix} \Bigg/ = \lim\limits_{t\to 0} \dfrac{1-\cos 2t}{t\ln(3t+1)} =$

$$= \lim_{t\to 0} \frac{2\sin^2 t}{t\ln(3t+1)} = \lim_{t\to 0} 2\cdot \underbrace{\frac{\sin^2 t}{t^2}}_{\to 1} \cdot \underbrace{\frac{3t}{\ln(3t+1)}}_{\to 1} \cdot \frac{1}{3} = \frac{2}{3}$$

Svar: a) $\frac{2}{3}$ b) $\frac{3}{2}$ c) $\frac{2}{3}$.

5 $z^5 + 4z^3 + 8z^2 + 32 = 0$ har en rot $z = ai$, $a\in\mathbb{R}$

Sätt $p(z) = z^5 + 4z^3 + 8z^2 + 32 \Rightarrow p(ai) = a^5 i - 4a^3 i - 8a^2 + 32 = 0$

$\Leftrightarrow (32-8a^2) + i(a^5 - 4a^3) = 0 \Leftrightarrow \begin{cases} 32-8a^2 = 0 \\ a^5 - 4a^3 = 0 \end{cases} \Leftrightarrow \begin{cases} a^2 = 4 \\ a^3(a^2-4) = 0 \end{cases}$

$\Leftrightarrow a = 2,\ a = -2$. Dvs $z_1 = 2i$, $z_2 = -2i$ – rötterna till $p(z) = 0$. Enligt Faktorsatsen får vi

$p(z) = (z-2i)(z+2i)\,q(z) = (z^2+4)\,q(z)$ där

$q(z) = \dfrac{p(z)}{z^2+4}$ $\overset{\text{Polynomdivision}}{\Rightarrow}$

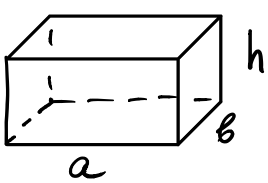
$$\begin{array}{rl} z^3 + 8 & \\ \hline z^5 + 4z^3 + 8z^2 + 32 & \big|\ z^2+4 \\ -\ (z^5 + 4z^3) & \\ \hline 8z^2 + 32 & \\ -\ (8z^2 + 32) & \\ \hline 0 & \end{array}$$

Alltså $p(z) = (z^2+4)(z^3+8) =$

$= (z^2+4)(z+2)(z^2-2z+4) =$

$= (z^2+4)(z+2)\left((z-1)^2+3\right) = (z^2+4)(z+2)(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)$

Svar: $z_{1,2} = \pm 2i$, $z_3 = -2$, $z_{4,5} = 1 \pm i\sqrt{3}$.

6 

$V = abh = 400\,dm^3$, $a = 2b$ $\Rightarrow$ $h = \dfrac{400}{2b^2} = \dfrac{200}{b^2}$

Materialkostnaden: $f(b) = 7\cdot(ab) + 5(2\cdot b\cdot h + 2\cdot a\cdot h + ab) =$

$7\cdot 2b\cdot b + 5\cdot\left(2\cdot b\cdot \frac{200}{b^2} + 2\cdot 2b\cdot\frac{200}{b^2} + 2b\cdot b\right) \Rightarrow$

$f(b) = 24b^2 + \dfrac{6000}{b}$, $b > 0$. Vi söker $\min\limits_{b>0} f(b)$

$f'(b) = 48b - \dfrac{6000}{b^2} = 0 \Leftrightarrow 48b^3 = 6 \cdot 10^3 \Leftrightarrow b = \dfrac{10^3}{8} =$

$= \dfrac{10^3}{2^3} = 5^3 \Leftrightarrow b = 5$

b	0		5		
f'		$-$		$+$	
f		$\searrow$	lok. min.	$\nearrow$	

I intervallet $]0,\infty[$ är f kontinuerlig $\Rightarrow$

$f_{min} = f(5)$ då $b = 5$ dm, $a = 10$ dm, $h = \dfrac{200}{25}$ dm $= 8$ dm

Svar: lådans mått är $a = 10$ dm, $b = 5$ dm, $h = 8$ dm.

7 För vilka a har $\dfrac{e^{-2x}}{x} = a$ exakt 2 lösningar?

Sätt $f(x) = \dfrac{e^{-2x}}{x}$. Vi ritar funktionskurvan $y = f(x)$.

- $D_f = \{x : x \neq 0\}$
- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{e^{-2x}}{x} = -\infty$ (enligt hastighetstabell).

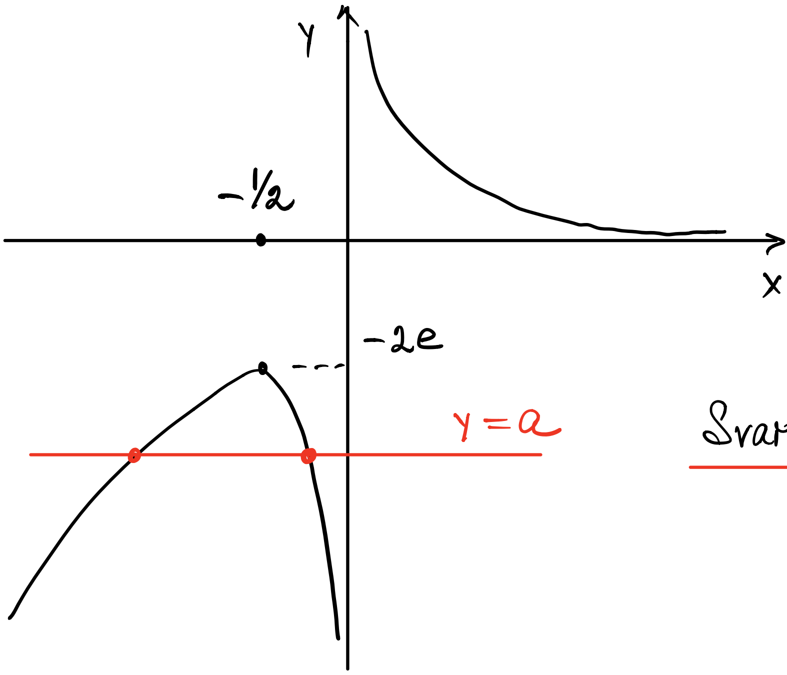
  $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^{-2x}}{x} = 0 \Rightarrow y = 0$ - vågrät asymptot

  $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{e^{-2x}}{x} = -\infty$, $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{e^{-2x}}{x} = \infty \Rightarrow$

  $x = 0$ - lodrät asymptot.
- $f'(x) = \dfrac{-2e^{-2x} \cdot x - e^{-2x}}{x^2} = -\dfrac{e^{-2x}}{x^2}(2x+1) = 0 \Leftrightarrow x = -\frac{1}{2}$

x		$-\frac{1}{2}$		0	
f'	$+$		$-$		$-$
f	$\nearrow$	lok. max	$\searrow$		$\searrow$

$f(-\frac{1}{2}) = -2e$



Svar: ekvationen $\frac{e^{-2x}}{x} = a$ har exakt 2 lösningar för $a < -2e$.